Formal Analysis: Canonical two-player games and optimization

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Week 1 Session 2

Classic games

Key terms

True or false: In a two-player game,

1. if both players have a **dominant strategy**, it is a **Nash equilibrium** for each to play this strategy.
2. both players must be playing a **dominant strategy** in any **Nash equilibrium**.
3. if each player's action is a **best response** to the other's, it must be a **Nash equilibrium.**

Are these equivalent games?

Kydd's version

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	3,3	1,4
	Defect	4,1	2,2

My version

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	3,3	1,100
	Defect	95,-1	2.1,2.6

Finding pure strategy Nash equilibria (PSNE) of 2-player games

Basic idea:

1. For each player, identify the **best response** to each (pure) **strategy** by other player
2. Identify **strategy profiles** in which both players' strategy is a best response

Optimization with the contest success function

Contest success function: simplest version

Common way to model how the outcome of a contest (war, election, litigation, etc) depends on the players' investments: **contest success function (CSF)**.

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Contest success function: simplest version

Common way to model how the outcome of a contest (war, election, litigation, etc) depends on the players' investments: **contest success function (CSF)**.

Let $p(m_1, m_2)$ be probability of 1 winning as function of investment made by 1 and 2 (m_1, m_2). Then

$$p(m_1, m_2) = \frac{m_1}{m_1 + m_2}$$

unless $m_1 = m_2 = 0$, in which case $p(m_1, m_2) = 1/2$.

Contest success function: more general version

Let $m = \{m_1, m_2, \ldots, m_k\}$ be vector of investments by k players, and let $p_i(m)$ be the probability of i winning.

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Let $m = \{m_1, m_2, \ldots, m_k\}$ be vector of investments by k players, and let $p_i(m)$ be the probability of i winning. Then

$$p_i(m) = \frac{f_i(m_i)}{\sum_{j=1}^{k} f_j(m_j)}$$

unless $m_i = 0$ for all i, in which case $p_i(m) = 1/k$ for all i.

Contest success function and player utility

Kydd 3.2.3 Two agents, 1 and 2, choosing military power m_1 and m_2.

For agent $i \in \{1, 2\}$, the utility function (Kydd p. 41) is

$$u_i = \frac{m_i}{m_1 + m_2} - \gamma_i m_i$$

Task 1: Explain/justify this utility function for the case of competition for power between states.

Task 2: Explain/justify this utility function for a different case: electoral competition between candidates. What do m_1, m_2, γ_i mean? Why might γ_1 and γ_2 differ?

Optimization: beginner level

A warlord is choosing a tax rate $\tau \in [0, 1]$ to apply to her subjects.

If the subjects produce output Y, the warlord collects τY.

Warlord's utility is linear in tax receipts: $U = \tau Y$.

Citizens' production depends on tax rates: $Y = (1 - \tau)L$.

What is the warlord's utility-maximizing tax rate?

See more in Ben Ansell's notes "Utility, optimization, and welfare (HT2017)" p. 10, on Slack

Optimization: beginner level (solution)

Considering how production depends on the tax rate, the warlord's utility is $U = \tau(1-\tau)L$.

First-order condition (FOC): In any maximum of this function, the first derivative of U w.r.t. τ must be zero.

First derivative with respect to τ:

$$\frac{\partial U}{\partial \tau} = (1-2\tau)L$$

The FOC is then $(1-2\tau*)L = 0$, so $\tau* = 1/2$.

Note: FOC is a *necessary* condition for a maximum. The *necessary and sufficient* condition is that the first derivative of U w.r.t. τ is zero **and** the second derivative of U w.r.t. τ is negative (**second-order condition (SOC)**).

$$\frac{\partial^2 U}{\partial \tau^2} = -2L$$

Optimization with the CSF: harder

Kydd 3.2.3 Two agents, 1 and 2, choosing military power m_1 and m_2.

For agent $i \in \{1, 2\}$, the utility function (Kydd p. 41) is

$$u_i = \frac{m_i}{m_1 + m + 2} - \gamma_i m_i$$

Task: find player 1's optimal level of military power as a function of player 2's military power.

Optimization with the CSF: solution

To get the **FOC**, first take the derivative of player 1's utility function with respect to m_1:

$$\frac{\partial u_1(m_1, m_2)}{\partial m_1} = \frac{m_1}{(m_1 + m_2)^2} - \gamma_1$$

This uses the quotient rule.

Setting this equal to 0 and rearranging, we get

$$m_1^* = \sqrt{\frac{m_2}{\gamma_1}} - m_2$$

Because the problem is symmetric, we know that

$$m_2^* = \sqrt{\frac{m_1}{\gamma_2}} - m_1$$

Should check SOC.

Optimization with the CSF: solution

To make things simpler, let's assume $\gamma_1 = \gamma_2 = \gamma$. (See Kydd for the more general solution.)

At the Nash equilibrium (if there is one), both players are maximizing their utility given the other's power. In other words, 1 is choosing the optimal m_1 given m_2, and 2 is choosing the optimal m_2 given m_1.

This means we can insert 2's optimal m_2 (which I write below as $m_2(m_1)$ to emphasize that it is a function of m_1) into 1's optimization problem.

$$
\begin{aligned}
m_1 &= \sqrt{\frac{m_2(m_1)}{\gamma}} - m_2(m_1) \\
m_1 &= \sqrt{\frac{\sqrt{\frac{m_1}{\gamma}} - m_1}{\gamma}} - \left(\sqrt{\frac{m_1}{\gamma}} - m_1\right) \\
\sqrt{\frac{m_1}{\gamma}} &= \sqrt{\frac{\sqrt{\frac{m_1}{\gamma}} - m_1}{\gamma}} \\
\frac{m_1}{\gamma} &= \frac{\sqrt{\frac{m_1}{\gamma}} - m_1}{\gamma} \\
m_1 &= \sqrt{\frac{m_1}{\gamma}} - m_1 \\
2m_1 &= \sqrt{\frac{m_1}{\gamma}} \\
(2m_1)^2 &= \frac{m_1}{\gamma} \\
m_1 &= \frac{1}{2\gamma}
\end{aligned}
$$

Because the problem is symmetric, $m_2 = \frac{1}{2\gamma}$.