Formal Analysis

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Plan for today

- What is formal theory for?
- Overview of the course & syllabus
- Preferences, rationality, utility, expected utility

What is formal theory for?

What are theories?

Theories are things we believe to be true, at least provisionally.

Theories are claims that make sense of regularities/patterns in the world.

What do we do with theories?

- Prediction
  - How would we expect X to affect Y?
- Explanation
  - X and Y are related. Why?

What's wrong with the theory in most dissertations?

- No theory
  - No reason to expect the predicted pattern, or any pattern
  - No explanation for the finding
- Bad theory
  - Conclusions don't follow from assumptions
  - Some assumptions not necessary
  - Theory doesn't address important features of problem
  - Assumptions not justified, too specific
- Poorly motivated theory
  - No puzzle to explain
  - Prediction obvious or produced by many theories

Theories and theory testing: physics

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We design a **crucial experiment** for which the two theories make different predictions.

We run the experiment and discard one of the theories.

Theories and theory testing: social science

Social science reality: The social world is very complicated, so our theories are probabilistic, scope-limited, partial → experiments rarely crucial. Theories rarely discarded.

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Physics theory: Elaborating theories to develop crucial experiments, keep only what works best

Social science theory: Elaborating theories to improve internal coherence, make judgments about what is useful

Why should theory be formal?

Doesn't have to be.

Why should theory be formal?

Doesn't have to be. But can be useful for

- Development of theory
  - Logical consistency
  - Abstraction $\rightarrow$ connections
- Communication of theory (depends on the crowd)
  - **Good**: abstraction and simplicity clarify and enlighten
  - **Bad**: notation and complexity overwhelm and confuse

Types of formal theory

- **Decision theory**: how agents optimize, e.g. partisan voter in event of scandal
- **Game theory**: how optimizing agents interact, e.g. politician and voter

Misconceptions about formal theory

People think:

- Formal theory necessarily assumes that agents are
  - selfish (vs. altruistic)
  - materialistic (vs. e.g. idealistic)
  - perfectly informed
  - capable of heroic computations
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Actually:

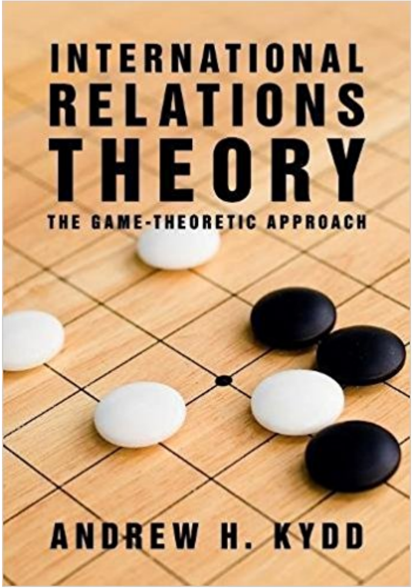
- Agents may be optimizing anything, and with constraints on info, processing power, etc
- "All models are wrong, but some are useful" (George Box)

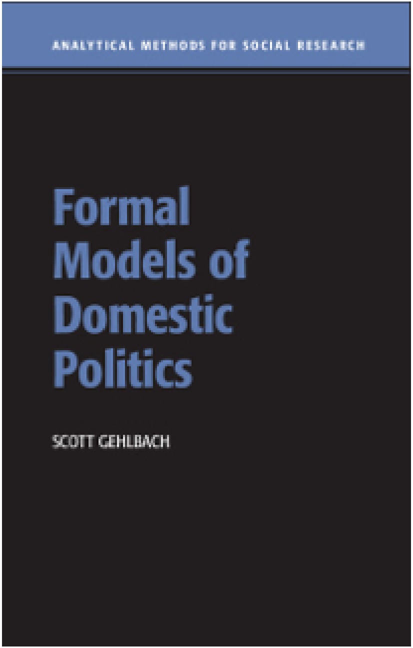
Pitfalls of formal theory

- Formal theorists like **interesting** theories, but sometimes the best theory is not interesting: e.g. domination.
- Modeling assumptions become beliefs about the world.
- They get carried away with technical challenges.

Overview of course

Textbooks


INTERNATIONAL
RELATIONS
THEORY
THE GAME-THEORETIC APPROACH
ANDREW H. KYDD


ANALYTICAL METHODS FOR SOCIAL RESEARCH
Formal
Models of
Domestic
Politics
SCOTT GEHLBACH

Schedule

Wk.Sess	Topic	Reading
1.1	Introduction and utility theory	Kydd 2
1.2	Strategic settings	Kydd 3
2.1	Bargaining	Kydd 4
2.2	Electoral competition under certainty	Gehlbach 1
3.1	Electoral competition under uncertainty	Gehlbach 2
3.2	Application (**Problem set 1 due**, January 31)	
4.1	Power change and war	Kydd 5
4.2	Private information and war	Kydd 6
5.1	Special interest politics	Gehlbach 3.1-3.4
5.2	Veto players	Gehlbach 4.1-4.4
6.1	Application (**Problem set 2 due**, February 19)	
6.2	Diplomacy and cheap talk	Kydd 9.1-9.3
7.1	Signaling	Kydd 9.4
7.2	Delegation 1	Gehlbach 5.1-5.4
8.1	Delegation 2	Gehlbach 5.5-5.7
8.2	Application (**Problem set 3 due**, March 7)	

Assessment

Applications and in-class quizzes	10%
Written assignment (due week 9)	20%
Problem sets (due weeks 3, 6, and 8)	30%
Final exam (take-home, due week 0 of TT)	40%

Background

This course will be harder if you've never seen notation like:

$$L = (p_1, \ldots, p_n)$$

$$U(L) \equiv \sum_{i=1}^{n} p_i u(x_i)$$

$$u(x) = x^2 \quad (1)$$

$$u'(x) = 2x \quad (2)$$

For catch-up/review, see Moore & Siegel's *A Mathematics Course for Political & Social Research* (2013) chapters 1, 2, 3, 5, 6, and 8.

Expectations

- You:
  - Do the reading for each session (at least attempt it).
  - If confused by reading, visit office hours (Tues, Thurs 11-12:30, Nuffield K4) and/or pose questions on Slack
  - If confused by what happens in class, stop me and/or do above
- I:
  - respond to your questions in office hours and on Slack
  - design quizzes, activities, mini-lectures, discussions, problem sets that reward your efforts in class and outside of class

Other business

- Need to reschedule meeting on 24th – 2-4pm on 25th okay?
- Piazza for questions, lecture notes, announcements

Preferences, rationality, utility, expected utility

Preferences

Define set of outcomes $X = \{a, b, c\}$.

- a at least as good as b: $a \succcurlyeq b$.
- a better than b: $a \succ b$.
- a no better or worse than b: $a \sim b$.

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If preferences are **complete**, $a \succcurlyeq b$ or $b \succcurlyeq a$ (or both) for any pair of alternatives a and b.

If preferences are **transitive**, $a \succcurlyeq b$ and $b \succcurlyeq c$ implies $a \succcurlyeq c$.

Rationality

If behavior is consistent with complete and transitive preferences, it is often called **rational**.

> *The theory of rational choice . . . is that a decision-maker chooses the best action according to her preferences, among all the actions available to her. No qualitative restriction is placed on the decision-maker's preferences; her rationality lies in the consistency of her decisions when faced with different sets of available actions, not in the nature of her likes and dislikes. (Osborne 2004, p. 4)*

Quick detour: if and only if, $\iff$, necessary and sufficient

Identical statements:

- **Condition A** is true if and only if **Condition B** is true
- Condition A $\iff$ Condition B
- A is a necessary and sufficient condition for B
- A is true whenever B is true and vice versa

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- Can a utility function be found for any set of preferences?
- Find a utility function for these preferences: $a \succ b$, $b \sim c$, $c \succ d$

Choice under uncertainty

Sometimes we are not choosing among outcomes $\{a, b, c\}$, but rather among actions $\{1, 2, 3\}$ that probabilistically lead to one of those outcomes

Examples: Voting, choosing a platform, challenging another state.

Each action leads to a *lottery* over alternatives.

Definition 2.3 (Kydd) A lottery associated with a finite set of outcomes, X, with number of elements equal to $|X| = n$, is a vector $L = (p_1, \ldots, p_n)$, where $p_i \in [0, 1]$ is interpreted as the probability that outcome i occurs, so that $\sum_i p_i = 1$.

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What is the expected utility of $L = \{.2, .3, .5\}$, given $u(x) = \{3, 2, 1\}$?

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But John von Neumann and Oscar Morgenstern (1947) proved that there is such a utility function if and only if preferences over lotteries are complete, transitive, continuous, and independent (**Theorem 2.2 in Kydd**).

Cardinal utilities, i.e. von Neumann Morgenstern (VNM) utilities

Morgenstern and Von Neumann, 1946

To show: if $u(x)$ is a VNM (cardinal) utility function (i.e. expected utility of lotteries tracks preferences over lotteries), then so is $a + bu(x)$, where $b > 0$.

What does this mean about cardinal utilities, in plain English?

Proof

To show: if $u(x)$ is a VNM (cardinal) utility function, then so is $a + bu(x)$, where $b > 0$.

Proof Call the expected utility of a lottery L under the original utility function $U(L)$, and call the expected utility a lottery L under the transformed utility function $V(L)$. We need to show that $U(L) \geq U(L') \iff V(L) \geq V(L')$ for all L, L'.

First we show $V(L) = a + bU(L)$. Recall $U(L) \equiv \sum_{i=1}^{n} p_i u(x_i)$. Observe that

$$V(L) \equiv \sum_{i=1}^{n} p_i \big(a + bu(x_i)\big) \quad (3)$$

$$= a \sum_{i=1}^{n} p_i + b \sum_{i=1}^{n} p_i u(x_i) \quad (4)$$

$$= a + bU(L). \quad (5)$$

Now, suppose that $U(L) \geq U(L')$ for some L, L'. Then $bU(L) \geq bU(L')$, assuming $b > 0$. And $a + bU(L) \geq a + bU(L')$ for any a. This implies that $U(L) \geq U(L') \iff V(L) \geq V(L')$. QED.

Cardinal utilities: relative values matter, but not scale or location

With **cardinal** utilities,

- the relative values matter: if $\{3, 2, 1\}$ works as cardinal utility, $\{3, 2, -1000\}$ does not.
- the scale does not matter: if $\{3, 2, 1\}$ works as cardinal utility, $\{300, 200, 100\}$ and $\{5, 4, 3\}$ and $\{2, 1, 0\}$ do too.

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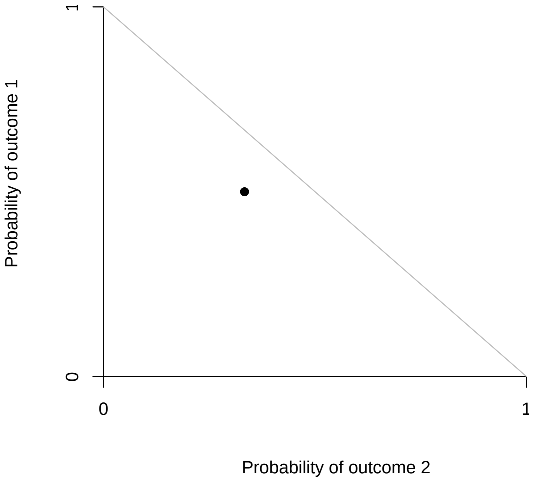
So we can e.g. **normalize** to 0-1 scale.

Intuitive sense of VNM's theorem (1)

Suppose there are only three outcomes.

Lotteries can be depicted as points on the **simplex**:



Intuitive sense of VNM's theorem (2)

Definition: Preferences over lotteries are *independent* if

$$L \succcurlyeq L^{'} \iff \alpha L + (1-\alpha)L^{''} \succcurlyeq \alpha L^{'} + (1-\alpha)L^{''}$$

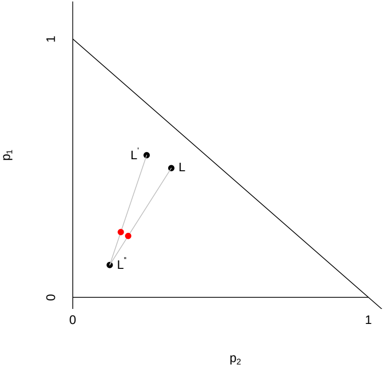
Kydd p. 15: "If $L_1 \succ L_2$, then adding an equal chance of obtaining L_3 to both sides does not alter the preference."

Intuitive sense of VNM's theorem (3a)

By definition, if preferences over lotteries are independent, then

$$L \sim L^{'} \iff \alpha L + (1-\alpha)L^{''} \sim \alpha L^{'} + (1-\alpha)L^{''}$$

Then also indifferent between lotteries at the red points below.

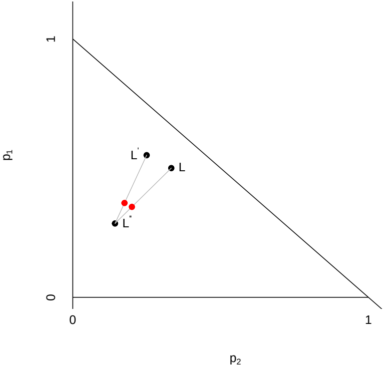


Intuitive sense of VNM's theorem (3b)

By definition, if preferences over lotteries are independent, then

$$L \sim L' \iff \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''$$

Also indifferent between lotteries at the red points below.



Intuitive sense of VNM's theorem (3c)

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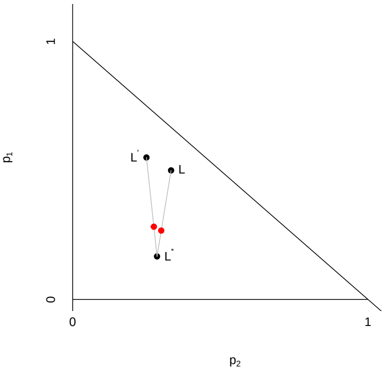
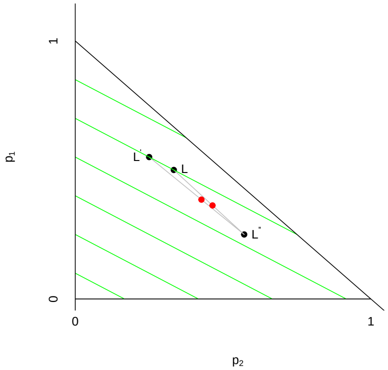


Intuitive sense of VNM's theorem (3d)

By definition, if preferences over lotteries are independent, then

$$L \sim L^{'} \iff \alpha L + (1-\alpha)L^{''} \sim \alpha L^{'} + (1-\alpha)L^{''}$$

Then **indifference curves** are lines parallel to the line connecting L and $L^{'}$:



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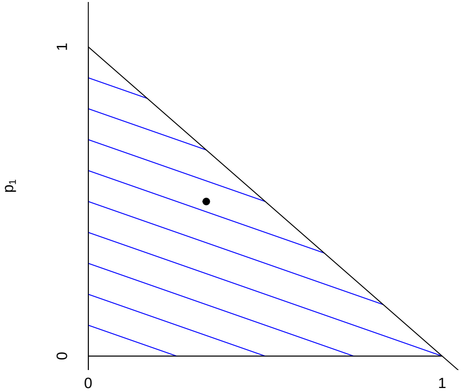
Then $EU(L) = p_1 + p_2 u_2$.

Rearranging, $p_1 = EU(L) - p_2 u_2$. We can thus plot a line connecting lotteries that have the same **expected utility** (an **isoquant**) for various values of $U(L)$:



What have we shown?

- If preferences are independent, **indifference curves** are parallel lines on the simplex
- **Isoquants of expected utility** are also parallel lines on the simplex; lines' slope depends on u_2
- So if preferences are independent then u_2 can be chosen so that indifference curves and isoquants of expected utility are **the same**, i.e. so that expected utility tracks preferences over lotteries.

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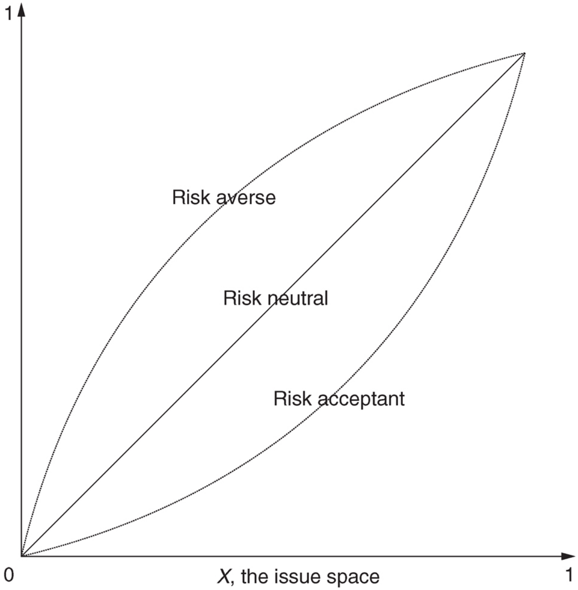
Google "Jonathan Levin choice under uncertainty" for more rigorous version.

Risk preferences



Other topics

- Formalizing strategic voting
- Two-dimensional preferences and bargaining