Formal Analysis

Andy Eggers

1/13/2018

Plan for today

- What is formal theory for?

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- Overview of the course & syllabus

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- Overview of the course & syllabus
- Preferences, rationality, utility, expected utility

What is formal theory for?

What are theories?

Theories are things we believe to be true, at least provisionally.

Theories are claims that make sense of regularities/patterns in the world.

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  - X and Y are related. Why?

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  - No puzzle to explain
  - Prediction obvious or produced by many theories

Theories and theory testing: physics

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We run the experiment and discard one of the theories.

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Social science reality: The social world is very complicated, so our theories are probabilistic, scope-limited, partial $\rightarrow$ experiments rarely crucial. Theories rarely discarded.

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Social science theory: Elaborating theories to improve internal coherence, make judgments about what is useful

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  - **Bad**: notation and complexity overwhelm and confuse

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- **Game theory**: how optimizing agents interact, e.g. politician and voter

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Actually:

- Agents may be optimizing anything, and with constraints on info, processing power, etc
- "All models are wrong, but some are useful" (George Box)

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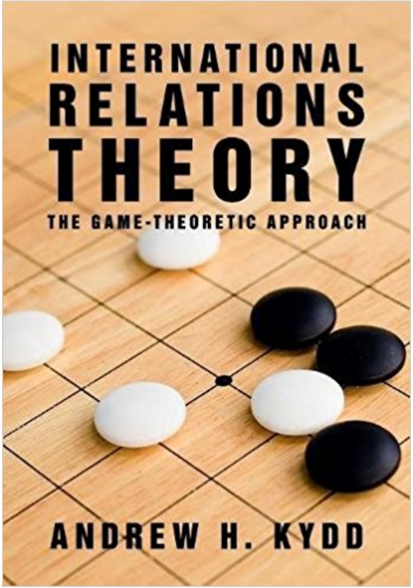
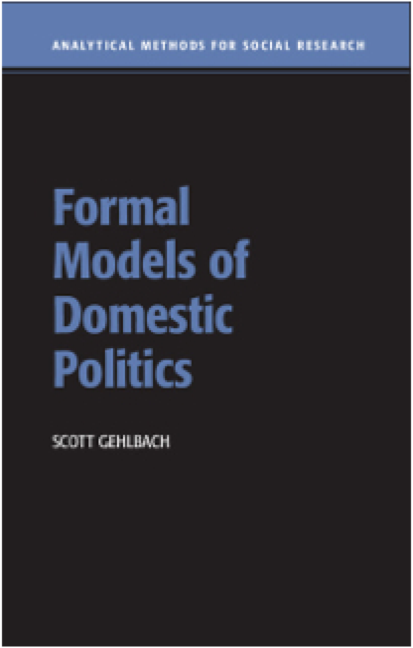
- Modeling assumptions become beliefs about the world.
- They get carried away with technical challenges.

Overview of course

Textbooks


INTERNATIONAL
RELATIONS
THEORY
THE GAME-THEORETIC APPROACH
ANDREW H. KYDD


ANALYTICAL METHODS FOR SOCIAL RESEARCH
Formal
Models of
Domestic
Politics
SCOTT GEHLBACH

Schedule

Wk.Sess	Topic	Reading
1.1	Introduction and utility theory	Kydd 2
1.2	Strategic settings	Kydd 3
2.1	Bargaining	Kydd 4
2.2	Electoral competition under certainty	Gehlbach 1
3.1	Electoral competition under uncertainty	Gehlbach 2
3.2	Application (**Problem set 1 due**, February 1)	
4.1	Power change and war	Kydd 5
4.2	Private information and war	Kydd 6
5.1	Arms competition and war	Kydd 7
5.2	Cooperation theory	Kydd 8
6.1	Application (**Problem set 2 due**, February 20)	
6.2	Special interest politics	Gehlbach 3.1-3.4
7.1	Veto players	Gehlbach 4
7.2	Delegation	Gehlbach 5
8.1	Diplomacy and signaling	Kydd 9
8.2	Application (**Problem set 3 due**, March 8)	

Assessment

Applications and in-class quizzes	10%
Written assignment (due week 9)	20%
Problem sets (due weeks 3, 6, and 8)	30%
Final exam (take-home, due week 0 of TT)	40%

Background

This course will be harder if you've never seen notation like:

$$L = (p_1, \ldots, p_n)$$

$$U(L) \equiv \sum_{i=1}^{n} p_i u(x_i)$$

$$u(x) = x^2 \quad (1)$$

$$u'(x) = 2x \quad (2)$$

For catch-up/review, see Moore & Siegel's *A Mathematics Course for Political & Social Research* (2013) chapters 1, 2, 3, 5, 6, and 8.

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- I:
  - respond to your questions in office hours and on Slack
  - design quizzes, activities, mini-lectures, discussions, problem sets that reward your efforts in class and outside of class

Preferences, rationality, utility, expected utility

Preferences

Define set of outcomes $X = \{a, b, c\}$.

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If preferences are **complete**, $a \succcurlyeq b$ or $b \succcurlyeq a$ (or both) for any pair of alternatives a and b.

If preferences are **transitive**, $a \succcurlyeq b$ and $b \succcurlyeq c$ implies $a \succcurlyeq c$.

Rationality

If behavior is consistent with complete and transitive preferences, it is often called **rational**.

> *The theory of rational choice ... is that a decision-maker chooses the best action according to her preferences, among all the actions available to her. No qualitative restriction is placed on the decision-maker's preferences; her rationality lies in the consistency of her decisions when faced with different sets of available actions, not in the nature of her likes and dislikes. (Osborne 2004, p. 4)*

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- Condition A $\iff$ Condition B
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- A is true whenever B is true and vice versa

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- Can a utility function be found for any set of preferences?
- Find a utility function for these preferences: $a \succ b$, $b \sim c$, $c \succ d$

Choice under uncertainty

Sometimes we are not choosing among outcomes $\{a, b, c\}$, but rather among actions $\{1, 2, 3\}$ that probabilistically lead to one of those outcomes

Example: Voting.

Each action leads to a *lottery* over alternatives.

Definition 2.3 (Kydd) A lottery associated with a finite set of outcomes, X, with number of elements equal to $|X| = n$, is a vector $L = (p_1, \ldots, p_n)$, where $p_i \in [0, 1]$ is interpreted as the probability that outcome i occurs, so that $\sum_i p_i = 1$.

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What is the expected utility of $L = \{.2, .3, .5\}$, given $u(x) = \{3, 2, 1\}$?

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This can't be true for all imaginable preferences.

But John von Neumann and Oscar Morgenstern (1947) proved that there is such a utility function if and only if preferences over lotteries are complete, transitive, continuous, and independent (**Theorem 2.2 in Kydd**).

Cardinal utilities, i.e. von Neumann Morgenstern (VNM) utilities

Morgenstern and Von Neumann, 1946

To show: if $u(x)$ is a VNM (cardinal) utility function (i.e. expected utility of lotteries tracks preferences over lotteries), then so is $a + bu(x)$, where $b > 0$.

What does this mean about cardinal utilities, in plain English?

Proof

To show: if $u(x)$ is a VNM (cardinal) utility function, then so is $a + bu(x)$, where $b > 0$.

Proof Call the expected utility of a lottery L under the original utility function $U(L)$, and call the expected utility a lottery L under the transformed utility function $V(L)$. We need to show that $U(L) \geq U(L') \iff V(L) \geq V(L')$ for all L, L'.

First we show $V(L) = a + bU(L)$. Recall $U(L) \equiv \sum_{i=1}^{n} p_i u(x_i)$. Observe that

$$V(L) \equiv \sum_{i=1}^{n} p_i \big(a + bu(x_i)\big) \tag{3}$$

$$= a \sum_{i=1}^{n} p_i + b \sum_{i=1}^{n} p_i u(x_i) \tag{4}$$

$$= a + bU(L). \tag{5}$$

Now, suppose that $U(L) \geq U(L')$ for some L, L'. Then $bU(L) \geq bU(L')$, assuming $b > 0$. And $a + bU(L) \geq a + bU(L')$ for any a. This implies that $U(L) \geq U(L') \iff V(L) \geq V(L')$. QED.

Cardinal utilities: relative values matter, but not scale or location

With **cardinal** utilities,

- the relative values matter: if $\{3, 2, 1\}$ works as cardinal utility, $\{3, 2, -1000\}$ does not.
- the scale does not matter: if $\{3, 2, 1\}$ works as cardinal utility, $\{300, 200, 100\}$ and $\{5, 4, 3\}$ and $\{2, 1, 0\}$ do too.

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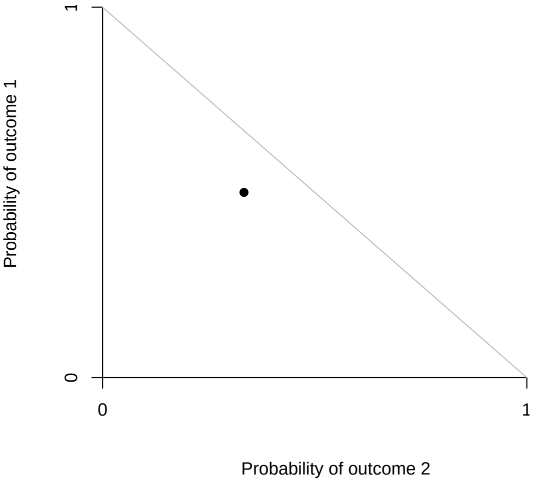
So we can e.g. **normalize** to 0-1 scale.

Intuitive sense of VNM's theorem (1)

Suppose there are only three outcomes.

Lotteries can be depicted as points on the **simplex**:



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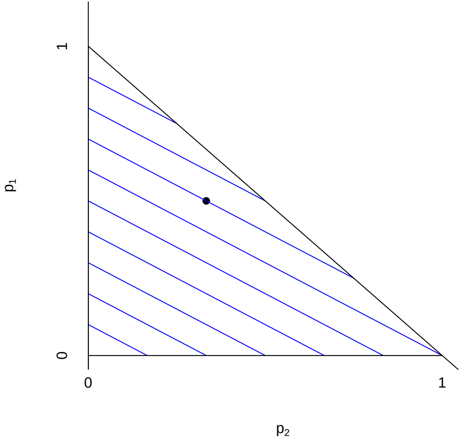
Rearranging, $p_1 = U(L) - p_2 u_2$. We can plot this line for different values of $U(L)$:



Intuitive sense of VNM's theorem (3)

Definition: Preferences over lotteries are *independent* if

$$L \succcurlyeq L^{'} \iff \alpha L + (1-\alpha)L^{''} \succcurlyeq \alpha L^{'} + (1-\alpha)L^{''}$$

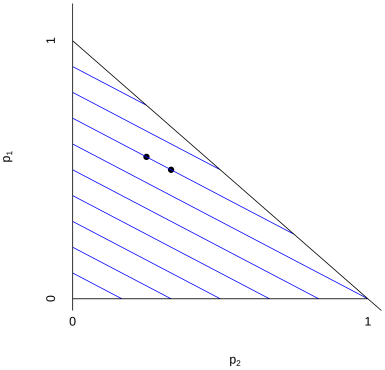
Kydd p. 15: "If $L_1 \succ L_2$, then adding an equal chance of obtaining L_3 to both sides does not alter the preference."

Intuitive sense of VNM's theorem (4)

Independence implies that

$$L \sim L' \iff \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''$$

And this implies that the **indifference curves** follow the same pattern as the expected utilities. Therefore preferences that satisfy the axioms can be represented by expected utility.



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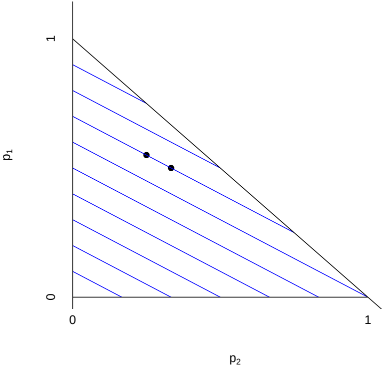

Google "Jonathan Levin choice under uncertainty" for more rigorous version.

Risk preferences